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2005 J. Phys. A: Math. Gen. 38 10917

(http://iopscience.iop.org/0305-4470/38/50/004)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 10917–10934 doi:10.1088/0305-4470/38/50/004

Genus-*N* algebraic reductions of the Benney hierarchy within a Schottky model

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Received 20 July 2005, in final form 23 October 2005
Published 30 November 2005
Online at stacks.iop.org/JPhysA/38/10917

Abstract

By exploiting a new theoretical connection between reductions of the Benney hierarchy and the Dirichlet problem for Laplace's equation, the solution to a spectral problem associated with *N*-parameter algebraic reductions of the Benney hierarchy is found explicitly. The solutions can be written in terms of the modified Green's function associated with reflectionally symmetric, *N*-connected planar domains whose 'holes' are all centred on the symmetry axis. Explicit formulae for the modified Green's function in a canonical class of circular domains are constructed using a Schottky model of the Schottky double of these domains. Uniformizations of the spectral problem associated with two different types of reductions then follow from these formulae.

PACS numbers: 02.30.Ik, 02.30.Jr, 02.30.Zz

1. Introduction

It is now known that there are important theoretical connections between integrability and integrable hierarchies and the Dirichlet problem for Laplace's equation in two-dimensional planar domains [30, 31, 33]. Consequently, the first-type Green's function for the Dirichlet boundary value problem for Laplace's equation in planar domains now has an important role in the mathematics of the dispersionless Toda hierarchy and the universal Whitham hierarchy.

The present author [3] has recently pointed out the significance of the Dirichlet boundary value problem (and its associated Green's functions) for another integrable hierarchy, namely the Benney hierarchy (or Benney moment equations) [1]. Since Gibbons and Tsarev [24] showed how a class of finite reductions of the Benney moment equations can be described in terms of conformal slit mappings, much effort has been invested in explicitly constructing such solutions. The elliptic case was studied in detail by Yu and Gibbons [25], while Baldwin and Gibbons [26, 27] have recently made progress on the hyperelliptic case. In all cases, the uniformization of two spectral functions associated with the finite reductions is performed on the Jacobi variety of the associated algebraic curve and the final results are expressed in terms of Riemann theta functions (or, equivalently, Kleinian sigma functions).

The new contribution of [3] is to show that a natural way to uniformize two spectral functions arising in these reductions is to use the Schottky double associated with reflectionally symmetric, N-connected planar domains, the holes of which are centred on the axis of symmetry. In such a domain, a mathematical object known as a *modified Green's function* can be defined. It satisfies a boundary value problem that is dual to that satisfied by the usual first-type Green's function in the domain. Formulae for two special types of N-parameter reductions of the Benney hierarchy can be written in terms of the analytic extension of this modified Green's function.

The general formulae derived in [3] are significant because both the first-type Green's function and the modified Green's function associated with a given planar domain can be written in terms of the prime form on its Schottky double. Explicit formulae for the Green's function are given, for example, in [32] where they are represented using Riemann theta functions. Analogous formulae exist for the modified Green's function. It should be clear that any such formulae, together with the new formulae derived in [3] expressing the reductions of the Benney hierarchy in terms of the modified Green's function, can be combined to give explicit general formulae for the required uniformizations of the spectral functions.

This paper carries out this programme in practice. Here, instead of performing the analysis on the Jacobi variety of the associated algebraic curves, we elect to employ a Schottky model of the Schottky double associated with a suitable N-connected domain. This model realizes the compact Riemann surface as a quotient space of the extended complex plane under the action of a classical Schottky group with $2(N-1)$ generators (for an N-connected domain). The generators are Möbius maps identifying pairs of circles in the plane. The domains considered here are all *circular domains*—that is, domains whose boundary components are all circular. Explicit formulae for the modified Green's function associated with these N-connected circular domains are found in terms of a special transcendental function, the *Schottky–Klein prime function* [10], associated with a Schottky group generated by a set of Möbius maps taking the enclosed circular boundaries to their reflections in the outer circular boundary of the domain.

Belokolos *et al* [13] discuss Schottky uniformizations of algebraic curves in respect of other well-known integrable systems of equations. The work here exhibits a novel application of the Schottky uniformization to reductions of the Benney hierarchy. The modified Green's function plays a pivotal role in producing the formulae.

2. The Benney hierarchy

The Benney equations [1] are

$$\begin{aligned} &u_t + uu_x - \left(\int_0^y u_x(x, y', t)\,\mathrm{d}y'\right) u_y + h_x = 0, \\ &h_t + uh_x + \left(\int_0^h u_x(x, y', t)\,\mathrm{d}y'\right) u_y = 0. \end{aligned} \tag{1}$$

Benney showed that if moments $A_n(x, t)$ are defined by

$$A_n(x, t) = \int_0^h u^n\,\mathrm{d}y, \tag{2}$$

then they satisfy the infinite set of equations

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + nA_{n-1}\frac{\partial A_0}{\partial x} = 0, \qquad n = 1, 2, \ldots, \tag{3}$$

which are the *Benney moment equations*. An identical set of moment equations can be derived from a Vlasov equation

$$\frac{\partial f}{\partial t_2} + p\frac{\partial f}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial f}{\partial p} = 0, \tag{4}$$

where $f(x, p, t)$ is some distribution function and the moments are now defined as

$$A_n = \int_{-\infty}^{\infty} p^n f \,\mathrm{d}p. \tag{5}$$

Benney [1] showed that the moment equations have an infinite number of conserved densities which are polynomial in the moments $\{A_n|n = 0, 1, \ldots\}$.

The best way to see this is to consider the generating function of the moments defined by

$$\lambda_R(x, p, t) = p + \sum_{n=0}^{\infty} \frac{A_n(x, t)}{p^{n+1}}, \tag{6}$$

which is the asymptotic series as $p \to \infty$ of

$$p + \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x, p', t)}{p - p'}\,\mathrm{d}p'. \tag{7}$$

The notation $\mathcal{P}\int$ denotes the principal-value integral. Gibbons and Tsarev [24] have found a method for constructing a family of solutions to the above equations. Their method is to define

$$\lambda(p) = p + \int_{\Lambda} \frac{f(x, p', t)}{p - p'}\,\mathrm{d}p', \tag{8}$$

where Λ is an indented contour passing below the point p on the real p'-axis. This function has the same large-p asymptotics as the function defined in (7) but, importantly, it can be analytically continued into the upper-half p-plane. Gibbons and Tsarev [24] have shown that N-parameter reductions of the integral equation (8) correspond to conformal slit mappings from an upper-half p-plane to an upper-half λ-plane having a collection of N non-intersecting slits emanating from fixed points on the real λ-axis into the upper-half λ-plane. Let $\{c_j|j = 1, \ldots, N\}$ be some fixed choice of Jordan arcs into the upper-half λ-plane from some set of fixed points $\{\lambda_0^{(j)}|j = 1, \ldots, N\}$ on the real λ-axis. The N-parameter reductions correspond to conformal mappings from the upper-half p-plane to a collection of N slits taken along these arcs and having end-points at some set of points $\{\hat{\lambda}_j|j = 1, \ldots, N\}$ on these arcs. These points are the Riemann invariants of the system and they have characteristic speeds $\hat{p}_i = p(\hat{\lambda}_i)$. In this way, construction of analytical forms for $\lambda(p)$ corresponds to being able to construct analytical formulae for such slit maps. It is this construction that will be the focus of the remainder of this paper.

3. Two types of N-parameter reduction

We now summarize the key formulae derived in [3]. Two types of N-parameter reduction were considered, each corresponding to a different class of arcs $\{c_j|j = 1, \ldots, N\}$ discussed in section 2.

Consider an arbitrary bounded N-connected domain D_z in a complex z-plane assumed to be reflectionally symmetric about the real z-axis. It is also taken that any holes in the domain are centred on the real z-axis. Let $G_0(z; z_\alpha)$ be the modified Green's function associated with D_z and with logarithmic singularity at z_α. A precise definition of the modified Green's function is given in [3]: it is the (unique, real) function having a single logarithmic singularity at a point z_α inside D_z that is constant on all the boundaries of D_z and satisfies the additional conditions that the integrals, with respect to arc length, of its normal derivatives around the boundary of D_z are all zero. Since $G_0(z; z_\alpha)$ is a harmonic function of z in D_z (except for the single logarithmic singularity at $z = z_\alpha$), we define $\tilde{G}_0(z; z_\alpha)$ to be its analytic extension, obtained by adding to $G_0(z; z_\alpha)$ its harmonic conjugate function $H_0(z; z_\alpha)$, i.e.,

$$\tilde{G}_0(z; z_\alpha) = G_0(z; z_\alpha) + \mathrm{i}H_0(z; z_\alpha). \tag{9}$$

For the first-type reductions, the formulae are

$$p(z) = \left[\frac{\partial}{\partial \bar{z}_\alpha} - \frac{\partial}{\partial z_\alpha}\right]\tilde{G}_0(z; z_\alpha), \qquad \lambda(z) = -\left[\frac{\partial}{\partial \bar{z}_\alpha} + \frac{\partial}{\partial z_\alpha}\right]\tilde{G}_0(z; z_\alpha) + C. \tag{10}$$

In this case, the arcs in the λ-plane correspond to vertical straight line segments. The above formulae can also be multiplied by a real constant. For the second type, the formulae are

$$\begin{aligned} p(z) &= \left[\frac{\partial}{\partial \bar{z}_\alpha} - \frac{\partial}{\partial z_\alpha}\right]\tilde{G}_0(z; z_\alpha), \\ \lambda(z) &= A\exp[\tilde{G}_0(z; z_\beta) - \tilde{G}_0(z; z_\alpha)] + C. \end{aligned} \tag{11}$$

This time, the arcs are a set of concentric circular arcs. In (10) and (11), z_α and z_β are points on the real z-axis inside D_z. Real constants A and C are chosen to ensure that

$$\lambda \sim p + \mathcal{O}(p^{-1}), \qquad \text{as} \quad z \to z_\alpha. \tag{12}$$

The importance of (10) and (11) is that formulae for the required spectral functions $p(z)$ and $\lambda(z)$ follow directly from knowledge of the modified Green's function $G_0(z; z_\alpha)$. It is the latter functions which we will now construct in the $N > 2$ case.

4. Circular domains and Schottky groups

The boundary value problem satisfied by the modified Green's function is conformally invariant. It is therefore enough to consider the solutions associated with a canonical class of reflectionally symmetric planar domains: a conformally equivalent choice of domain just corresponds to reparametrizations of the uniformizing variable. Therefore, consider a multiply connected, reflectionally symmetric *circular domain* with all its holes centred on the real axis. This is a finitely connected bounded domain all of whose boundary components are circles. Since we have now fixed a special class of domains, henceforth the notation D_ζ will be used to denote such a domain in a complex ζ-plane. Specifically, let D_ζ be the unit ζ-disc with $N-1$ smaller circular discs excised, all centred on the real ζ-axis. Let the circular boundaries of these smaller discs be denoted by $\{C_j \,|\, j = 1, \ldots, N-1\}$ and denote the outer unit circle $|\zeta| = 1$ by C_0. A definition sketch illustrating a quadruply connected circular domain is shown in figure 1. To uniquely specify such an N-connected domain D_ζ, only the centres and radii of the N enclosed circular boundaries are needed. Let the real numbers $\{\delta_j \,|\, j = 1, \ldots, N-1\}$ be their centres and let $\{q_j \,|\, j = 1, \ldots, N-1\}$ be their radii. For convenience, let $M = N - 1$.

To proceed with the construction, first define M Möbius maps $\{\phi_j \,|\, j = 1, \ldots, M\}$ corresponding to the conjugation map for points on the circle C_j. That is, if C_j has equation

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\bar{\zeta} - \delta_j) = q_j^2, \tag{13}$$

then

$$\bar{\zeta} = \delta_j + \frac{q_j^2}{\zeta - \delta_j}, \tag{14}$$

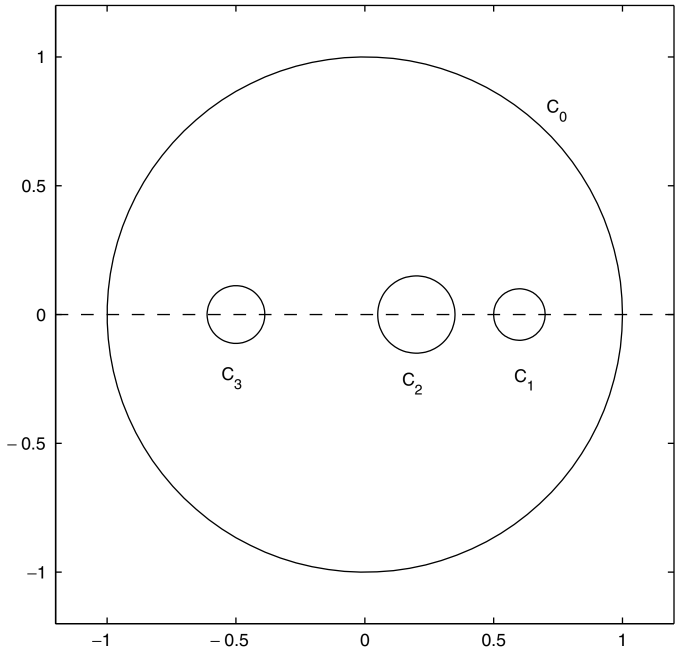

Figure 1. A typical quadruply connected reflectionally symmetric circular domain consisting of the unit ζ-disc with three enclosed circular holes centred on the real axis.

and so

$$\phi_j(\zeta) \equiv \delta_j + \frac{q_j^2}{\zeta - \delta_j}. \tag{15}$$

If ζ is a point on C_j, its complex conjugate is $\bar{\zeta} = \phi_j(\zeta)$. We will also define the map $\phi_0(\zeta) = 1/\zeta$.

Next, introduce the Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{q_j^2\zeta}{1 - \delta_j\zeta}. \tag{16}$$

Let C_j' be the circle obtained by reflection of the circle C_j in the unit circle $|\zeta| = 1$ (i.e., the circle obtained by the transformation $\zeta \mapsto \frac{1}{\zeta}$). It is easy to verify that the image of C_j' under the transformation θ_j is C_j. Thus, by virtue of these maps, we are equipped with a holomorphic identification between points on the circles C_j and C_j'. Since the M circles $\{C_j\}$ are non-overlapping, so are $\{C_j'\}$. The (classical) *Schottky group* Θ is defined to be the infinite free group of mappings generated by compositions of the $2M$ basic Möbius maps $\{\theta_j | j = 1, \ldots, M\}$ and their inverses $\left\{\theta_j^{-1} \middle| j = 1, \ldots, M\right\}$ and including the identity map. Beardon [11] gives a general discussion of such groups. An accessible discussion of Schottky groups and their mathematical properties can also be found in a recent monograph by Mumford, Series and Wright [18].

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles $\{C_j\}$ and $\{C_j'\}$. This region is a *fundamental region* associated with the Schottky group generated by the Möbius maps $\{\theta_j | j = 1, \ldots, M\}$ and their inverses. This fundamental region can be understood as having two 'halves'—the half that is inside the unit circle but exterior to the

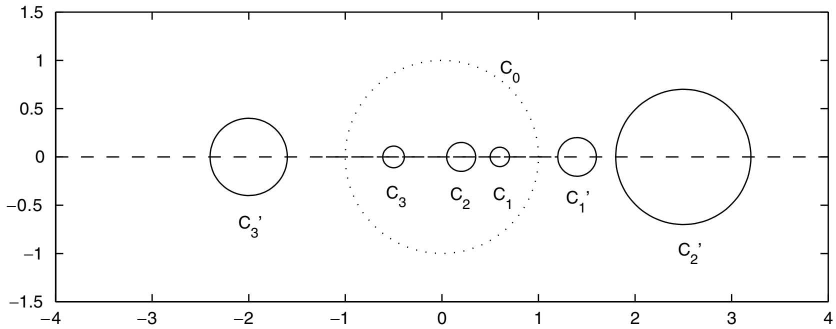

Figure 2. Schematic of the fundamental region associated with the reflectionally symmetric quadruply connected circular domain shown in figure 1. There are six Schottky circles C_1, C_1', C_2, C_2', C_3 and C_3'. The unbounded region exterior to all six circles is a fundamental region of the group.

circles C_j is the region we are calling D_ζ, the region outside the unit circle and exterior to the circles C_j' is the other half. A schematic illustrating this fundamental region and the Schottky circles associated with the quadruply connected domain of figure 1 is shown in figure 2.

There are two important properties of these Möbius maps that can easily be established. The first is that

$$\theta_j^{-1}(\zeta) = \frac{1}{\phi_j(\zeta)}, \qquad \forall \zeta. \tag{17}$$

This can be verified using definitions (15) and (16) (or, alternatively, by considering the geometrical effect of each map). The second property, following from the first, is that

$$\theta_j^{-1}(\zeta^{-1}) = \frac{1}{\phi_j(\zeta^{-1})} = \frac{1}{\overline{\bar{\phi}_j(\bar{\zeta}^{-1})}} = \frac{1}{\overline{\theta_j(\bar{\zeta})}} = \frac{1}{\bar{\theta}_j(\zeta)}, \qquad \forall \zeta. \tag{18}$$

Some special infinite subsets of mappings in a given Schottky group will be needed in what follows. A special notation is now introduced. This notation is not standard but is introduced here to clarify the presentation. The full Schottky group is denoted by Θ. The notation ${}_i\Theta_j$ is used to denote all mappings in the full group which do not have a power of θ_i or θ_i^{-1} on the left-hand end or a power of θ_j or θ_j^{-1} on the right-hand end. As a special case of this, the notation Θ_j simply means all mappings in the group which do not have any positive or negative power of θ_j on the right-hand end (but with no stipulation about what appears on the left-hand end). Similarly, ${}_j\Theta$ means all mappings which do not have any positive or negative power of θ_j on the left-hand end (but with no stipulation about what appears on the right-hand end). In addition, the single prime notation will be used to denote a subset where the identity is excluded from the set; thus, Θ_1' denotes all mappings, excluding the identity and all transformations with a positive or negative power of θ_1 on the right-hand end. The double prime notation will be used to denote a subset where the identity *and* all inverse mappings are excluded from the set. This means, for example, that if $\theta_1\theta_2$ is included in the set, the mapping $\theta_2^{-1}\theta_1^{-1}$ must be excluded. Thus, Θ'' means all mappings excluding the identity and all inverses. Similarly, the notation ${}_1\Theta_2''$ denotes all mappings, excluding inverses and the identity, which do not have any power of θ_1 or θ_1^{-1} on the left-hand end or any power of θ_2 or θ_2^{-1} on the right-hand end. In the same way, Θ_j'' denotes all mappings, excluding the identity and all inverses, which do not have any positive or negative power of θ_j on the right-hand end.

5. The Schottky–Klein prime function

Following Baker [10], the *Schottky–Klein prime function* is defined as

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\omega'(\zeta,\gamma), \tag{19}$$

where the function $\omega'(\zeta,\gamma)$ is given by

$$\omega'(\zeta,\gamma) = \prod_{\theta_i\in\Theta''} \frac{(\theta_i(\zeta)-\gamma)(\theta_i(\gamma)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\gamma)-\gamma)} \tag{20}$$

and where the product is over all mappings θ_i in the set Θ''. ω' can also be written as

$$\omega'(\zeta,\gamma) = \prod_{\theta_i\in\Theta''} \{\zeta,\theta_i(\zeta),\gamma,\theta_i(\gamma)\}, \tag{21}$$

where the brace notation denotes a cross ratio of the four arguments. This will be useful later. The function $\omega(\zeta,\gamma)$ is single valued on the whole ζ-plane, has a simple zero at γ and at all points equivalent to γ under the mappings of the group Θ. The prime notation is not used here to denote differentiation.

We proceed under the assumption that the choice of Schottky circles is such that the infinite product defining the prime function converges. The general question of convergence is a complicated matter. Baker [10] discusses some general criteria for convergence. Note that since the problem here demands that D_z be reflectionally symmetric about the real axis with all its holes centred thereon, the relevant Schottky groups turn out to be of the Fuchsian kind with an invariant circle. It is known that the Poincaré θ_2 series associated with such groups are convergent [12], while Burnside [14, 15] has shown how expression (19) for the prime function can be derived from such series. It should also be mentioned that, in cases involving Schottky groups of this kind, Crowdy and Marshall [4] have performed numerical comparisons of the uniformizations of certain algebraic curves using both the Schottky–Klein prime function *and* higher dimensional Poincaré theta series (see, in particular, the appendix of [4]). Numerically, excellent agreement is found between the two distinct uniformizations of a given algebraic curve. All this reassures us that the assumption of convergence of the infinite product is a reasonable one.

The Schottky–Klein prime function has some important transformation properties. One such property is that it is anti-symmetric in its arguments, i.e.,

$$\omega(\zeta,\gamma) = -\omega(\gamma,\zeta). \tag{22}$$

This is clear from inspection of (19) and (20). A second important property is given by

$$\frac{\omega(\theta_j(\zeta),\gamma_1)}{\omega(\theta_j(\zeta),\gamma_2)} = \beta_j(\gamma_1,\gamma_2)\frac{\omega(\zeta,\gamma_1)}{\omega(\zeta,\gamma_2)}, \tag{23}$$

where θ_j is any one of the basic maps of the Schottky group. A detailed derivation of this result is given in chapter 12 of [10]. A formula for $\beta_j(\gamma_1,\gamma_2)$ is

$$\beta_j(\gamma_1,\gamma_2) = \prod_{\theta_k\in\Theta_j} \frac{(\gamma_1-\theta_k(B_j))(\gamma_2-\theta_k(A_j))}{(\gamma_1-\theta_k(A_j))(\gamma_2-\theta_k(B_j))}, \tag{24}$$

where A_j and B_j are the two fixed points of the mapping θ_j satisfying

$$\theta_j(A_j) = A_j, \qquad \theta_j(B_j) = B_j. \tag{25}$$

They are therefore the two solutions of a quadratic equation. It follows that

$$\frac{\theta_j(\zeta)-B_j}{\theta_j(\zeta)-A_j} = \mu_j\,\mathrm{e}^{\mathrm{i}\kappa_j}\frac{\zeta-B_j}{\zeta-A_j} \tag{26}$$

for some real parameters μ_j and κ_j. The roots A_j and B_j are ordered such that $|\mu_j| < 1$ in (26). Note the property that

$$\beta_j(\gamma_1, \gamma_2) = \frac{1}{\beta_j(\gamma_2, \gamma_1)}. \tag{27}$$

A third property of $\omega(\zeta, \gamma)$ which will also be useful later is that, for the class of Schottky groups introduced earlier,

$$\bar{\omega}(\zeta^{-1}, \gamma^{-1}) = -\frac{1}{\zeta\gamma}\omega(\zeta, \gamma), \tag{28}$$

where the conjugate function $\overline{\omega}(\zeta, \gamma)$ is defined by

$$\overline{\omega}(\zeta, \gamma) = \overline{\omega(\bar{\zeta}, \bar{\gamma})}. \tag{29}$$

A detailed derivation of (28) is given in [5].

6. Explicit solution for G_j

Given a circular domain D_ζ, the associated Schottky–Klein prime function $\omega(\zeta, \gamma)$ can be constructed. Let the singularity of the modified Green's function $G_j(\zeta; \alpha)$ in D_ζ be at α and let it satisfy the conditions

$$\begin{aligned} &G_j(\zeta; \alpha) = 0, && \text{on} \quad C_j, \\ &G_j(\zeta; \alpha) = \gamma_{jk}(\alpha), && \text{on} \quad C_k, \quad k \neq j, \\ &\int_{C_k} \frac{\partial G_j(\zeta; \alpha)}{\partial n}\, \mathrm{d}s = 0, && k \neq j, \end{aligned} \tag{30}$$

where the set of parameters $\{\gamma_{jk}(\alpha)\}$ depends only on α, s denotes the arc length, while $\partial/\partial n$ denotes the normal derivative. Koebe [17] showed that the function satisfying these requirements is unique.

It will now be argued that an explicit expression for the required function is

$$G_j(\zeta; \alpha) = \frac{1}{2}\log\left|\frac{\omega(\zeta, \alpha)\overline{\omega}(\phi_j(\zeta), \phi_j(\alpha))}{\omega(\zeta, \overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta), \alpha)}\right|. \tag{31}$$

Note that since, on C_j,

$$\bar{\zeta} = \phi_j(\zeta), \tag{32}$$

the antiholomorphic involution taking a point α to its corresponding point on the Schottky double is

$$\alpha = \overline{\phi}_j(\bar{\alpha}). \tag{33}$$

Equation (31) therefore has logarithmic singularities at α and at its reflected point, $\overline{\phi}_j(\bar{\alpha})$, on the double.

First, define the $M + 1$ functions

$$R_j(\zeta; \alpha) = \frac{\omega(\zeta, \alpha)\overline{\omega}(\phi_j(\zeta), \phi_j(\alpha))}{\omega(\zeta, \overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta), \bar{\alpha})}, \tag{34}$$

where $j = 0, 1, \ldots, M$, so that

$$G_j(\zeta; \alpha) = \tfrac{1}{2}\log|R_j(\zeta; \alpha)|. \tag{35}$$

This means that $G_j(\zeta; \alpha)$ has a single isolated logarithmic singularity in D_ζ at $\zeta = \alpha$, as required. Since the zero of R_j at $\zeta = \alpha$ is second order, locally $G_j(\zeta; \alpha)$ has the expansion

$$G_j(\zeta; \alpha) = \log|\zeta - \alpha| + \mathcal{O}(1), \tag{36}$$

again as required.

It remains to verify that (31) satisfies the required boundary conditions on all the circles $\{C_j | j = 0, 1, \ldots, M\}$. It can be shown that, on circle C_k,

$$|R_j(\zeta;\alpha)| = \left| \frac{\beta_j(\overline{\phi_j}(\bar{\alpha}), \alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}), \alpha)} \right|. \tag{37}$$

As shown in the appendix, this formula holds for all integers j and k (between 0 and M) provided we adopt the convention that $\beta_0(\zeta, \alpha) \equiv 1$. It is immediate that, on C_j, $|R_j(\zeta, \alpha)| = 1$, so $G_j(\zeta, \alpha) = 0$ there. On C_k with $k \neq j$, we have

$$G_j(\zeta;\alpha) = \frac{1}{2} \log \left| \frac{\beta_j(\overline{\phi_j}(\bar{\alpha}), \alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}), \alpha)} \right|, \tag{38}$$

which gives explicit formulae for the constants $\{\gamma_{jk}(\alpha)\}$ defined in (30).

It is clear that the analytic extension $\tilde{G}_j(\zeta;\alpha)$ is given by

$$\tilde{G}_j(\zeta;\alpha) = \tfrac{1}{2} \log R_j(\zeta;\alpha). \tag{39}$$

Define the fundamental region associated with the function $\tilde{G}_j$ to be the domain D_ζ and its image under reflection in the jth circle. In this fundamental region, $\tilde{G}_j(\zeta, \alpha)$ has precisely two logarithmic singularities of equal and opposite strength: one at α and another at the reflected point $\overline{\phi}_j(\bar{\alpha})$. A natural way to define a branch of $\tilde{G}_j$ is therefore to join by a branch cut each such pair of logarithmic singularities in each copy of this fundamental region. It can then be argued that

$$\mathrm{Im}\left[\oint_{C_k} \mathrm{d}[\tilde{G}_j(\zeta;\alpha)] \right] = 0 \tag{40}$$

unless $k = j$ which, after a few manipulations, can be shown to be equivalent to the condition that the quantities

$$\oint_{C_k} \frac{\partial G_j}{\partial n} \, \mathrm{d}s \tag{41}$$

will be non-zero only for $k = j$.

Since we have identified a function satisfying all the conditions required of a modified Green's function, we now exploit the result of Koebe [17] which says that the latter function is unique. It follows that the functions $\{G_j(\zeta;\alpha)\}$ we have constructed are indeed the required set of modified Green's functions.

Although, for completeness, we have reported formulae for all the modified Green's functions $\{G_j | j = 0, 1, \ldots, N-1\}$, only $G_0(\zeta;\alpha)$ will be used in the analysis of the spectral problem for the Benney reductions. If preferred, any other choice of modified Green's function could also be used.

7. The elliptic reduction

Consider first the case of a doubly connected reflectionally symmetric circular domain D_ζ. Any doubly connected domain is conformally equivalent to an annular region $q < |\zeta| < 1$ for some value of the conformal modulus q [22]. D_ζ is therefore taken to be this annular domain.

7.1. First-type reduction

The first-type reduction is the same as that previously analysed by Yu and Gibbons [25]. This provides an instructive check on the preceding formulation. In this case, $\delta_1 = 0$ and $q_1 = q$, so that

$$\phi_1(\zeta) = \frac{q^2}{\zeta}, \qquad \theta_1(\zeta) = q^2\zeta. \tag{42}$$

The Schottky group is generated by θ_1 and its inverse: it is sometimes called the *loxodromic group*. Its elements are

$$\{\theta_1^j \,|\, j \in \mathbb{Z}\}. \tag{43}$$

From (19), the associated Schottky–Klein prime function can be shown to be

$$\omega(\zeta, \gamma) = -\frac{\gamma}{D^2} P(\zeta\gamma^{-1}), \tag{44}$$

where

$$P(\zeta) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-q^{2k}\zeta)(1-q^{2k}\zeta^{-1}) \tag{45}$$

and

$$D \equiv \prod_{k=1}^{\infty}(1-q^{2k}). \tag{46}$$

The modified Green's function $G_0(\zeta;\alpha)$ for this domain is

$$G_0(\zeta;\alpha) = \log\left| |\alpha| \frac{P(\zeta\alpha^{-1})}{P(\zeta\bar{\alpha})} \right|, \tag{47}$$

where α is a point inside D_ζ. It follows that the analytic extension of this function is

$$\tilde{G}_0(\zeta;\alpha) = \log\left(|\alpha| \frac{P(\zeta\alpha^{-1})}{P(\zeta\bar{\alpha})} \right). \tag{48}$$

This is the function used in [3].

Given (47), expressions for $p(\zeta)$ and $\lambda(\zeta)$ follow from the general formulae

$$p(\zeta) = \left[\frac{\partial}{\partial\bar{\alpha}} - \frac{\partial}{\partial\alpha}\right]\tilde{G}_0(\zeta;\alpha), \qquad \lambda(\zeta) = -\left[\frac{\partial}{\partial\bar{\alpha}} + \frac{\partial}{\partial\alpha}\right]\tilde{G}_0(\zeta;\alpha) + C \tag{49}$$

and letting α be real. Since

$$\frac{\partial\tilde{G}_0}{\partial\alpha} = \frac{1}{2\alpha} - \frac{1}{\alpha^2}\frac{\zeta P_\zeta(\zeta\alpha^{-1})}{P(\zeta\alpha^{-1})}, \qquad \frac{\partial\tilde{G}_0}{\partial\bar{\alpha}} = \frac{1}{2\bar{\alpha}} - \frac{\zeta P_\zeta(\zeta\bar{\alpha})}{P(\zeta\bar{\alpha})}, \tag{50}$$

where $P_\zeta(\zeta)$ denotes the derivative of $P(\zeta)$ with respect to ζ, it follows that

$$p(\zeta) = \frac{1}{\alpha}(K(\zeta\alpha^{-1}) - K(\zeta\alpha)), \qquad \lambda(\zeta) = \frac{1}{\alpha}(K(\zeta\alpha^{-1}) + K(\zeta\alpha) - 1) + C. \tag{51}$$

The function $K(\zeta)$ is defined as

$$K(\zeta) \equiv \frac{\zeta P_\zeta(\zeta)}{P(\zeta)}. \tag{52}$$

Formulae (51) give explicit expressions of the functions p and λ in terms of the uniformizing parameter ζ. The constant C is chosen to ensure that

$$\lambda \sim p + \mathcal{O}(p^{-1}), \qquad \text{as} \quad \zeta \to \alpha. \tag{53}$$

Straightforward algebra produces the result

$$C = \frac{1}{\alpha}(1 - 2K(\alpha^2)). \tag{54}$$

It was verified in [3] that these formulae are equivalent to those derived by Yu and Gibbons [25].

7.2. Second-type reduction

For the second-type reductions, on use of (48) in (11) it follows that

$$p(\zeta) = \frac{1}{\alpha}(K(\zeta\alpha^{-1}) - K(\zeta\alpha)), \qquad \lambda(\zeta) = \tilde{A}\frac{P(\zeta\beta^{-1})P(\zeta\alpha)}{P(\zeta\beta)P(\zeta\alpha^{-1})} + \tilde{C} \tag{55}$$

for some real constants $\tilde{A}$ and $\tilde{C}$ chosen so that

$$\lambda \sim p + \mathcal{O}(p^{-1}). \tag{56}$$

Defining $P'(\zeta)$ to be

$$P'(\zeta) = \prod_{k=1}^{\infty}(1 - q^{2k}\zeta)(1 - q^{2k}\zeta^{-1}) \tag{57}$$

so that

$$P(\zeta) = (1 - \zeta)P'(\zeta), \tag{58}$$

an expansion of (55) gives

$$\begin{aligned} \tilde{A} &= -\frac{P(\alpha\beta)P'(1)}{\alpha P(\alpha\beta^{-1})P(\alpha^2)}, \\ \tilde{C} &= \frac{1}{\alpha}(1 - K(\alpha^2)) + \alpha\tilde{A}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{P(\zeta\beta^{-1})P(\zeta\alpha)}{P(\zeta\beta)P'(\zeta\alpha^{-1})}\right)\Bigg|_{\zeta=\alpha}. \end{aligned} \tag{59}$$

8. Higher genus reductions

To demonstrate the efficacy of our construction, and how amenable the formulae are to numerical computation, some higher genus examples are now presented. To do this, we consider reflectionally symmetric circular domains of connectivity greater than 2. It follows from section 6 that

$$\tilde{G}_0(\zeta;\alpha) = \frac{1}{2}\log\left(\frac{\omega(\zeta,\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\bar{\alpha})}\right). \tag{60}$$

On use of (28), (60) simplifies to

$$\tilde{G}_0(\zeta;\alpha) = \log\left(\frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\bar{\alpha}^{-1})}\right). \tag{61}$$

For convenience, we introduce the abbreviated notation

$$\omega_\alpha(\zeta,\alpha) \equiv \frac{\partial}{\partial\alpha}\omega(\zeta,\alpha), \qquad \omega'_\alpha(\zeta,\alpha) \equiv \frac{\partial}{\partial\alpha}\omega'(\zeta,\alpha), \tag{62}$$

denoting derivatives of $\omega(\zeta,\alpha)$ and $\omega'(\zeta,\alpha)$ with respect to their second arguments. It follows that

$$\frac{\partial\tilde{G}_0}{\partial\alpha} = -\frac{1}{2\alpha} + \frac{\omega_\alpha(\zeta,\alpha)}{\omega(\zeta,\alpha)}, \qquad \frac{\partial\tilde{G}_0}{\partial\bar{\alpha}} = -\frac{1}{2\bar{\alpha}} + \frac{1}{\bar{\alpha}^2}\frac{\omega_\alpha(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})}. \tag{63}$$

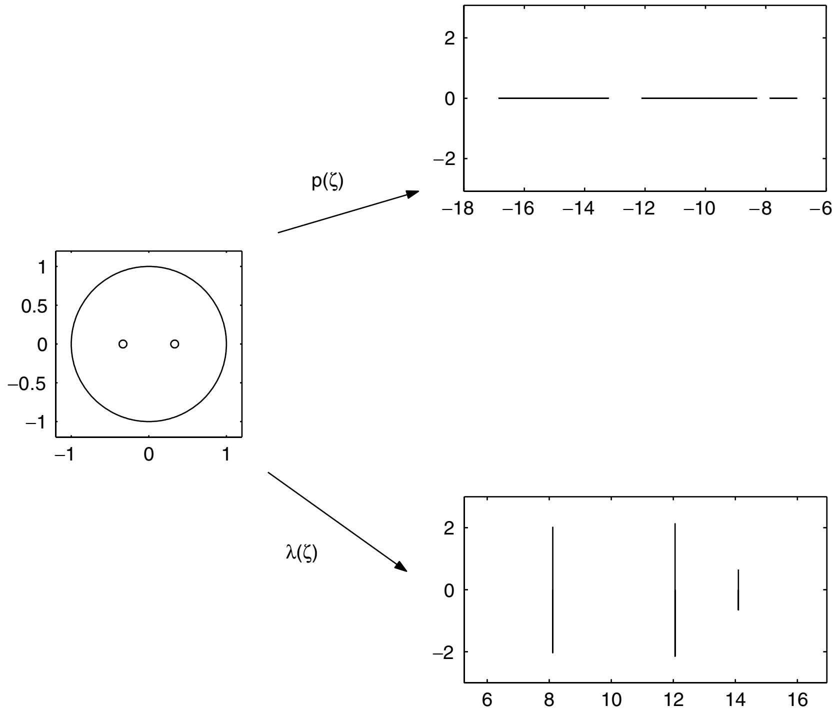

Figure 3. An illustration of the first type of reduction in the genus-2 case. The figures show the images of the annulus under the conformal mappings (64) with parameter values $\alpha = -0.1$, $q_1 = q_2 = 0.05$ and $\delta_1 = -\delta_2 = 1/3$.

8.1. First-type reductions

On use of (10) and (61), it follows that the solutions for $p(\zeta)$ and $\lambda(\zeta)$ for the first type of reduction are given by

$$
\begin{aligned}
p(\zeta) &= -\frac{1}{\alpha}\left(\alpha\frac{\omega_\alpha(\zeta,\alpha)}{\omega(\zeta,\alpha)} - \frac{1}{\alpha}\frac{\omega_\alpha(\zeta,\alpha^{-1})}{\omega(\zeta,\alpha^{-1})}\right),\\
\lambda(\zeta) &= -\frac{1}{\alpha}\left(\alpha\frac{\omega_\alpha(\zeta,\alpha)}{\omega(\zeta,\alpha)} + \frac{1}{\alpha}\frac{\omega_\alpha(\zeta,\alpha^{-1})}{\omega(\zeta,\alpha^{-1})} - 1\right) + C,
\end{aligned}
\tag{64}
$$

where we have now taken $\bar{\alpha} = \alpha$. Some algebra, from local expansions of the above functions about $\zeta = \alpha$, reveals that the real constant C is

$$
C = \frac{1}{\alpha}\left(\frac{2\omega_\alpha(\alpha,\alpha^{-1})}{\alpha\omega(\alpha,\alpha^{-1})} - 1\right). \tag{65}
$$

To illustrate these uniformizations, and to show how readily the formulae derived here can be computed in practice, the images of the circular domain D_ζ under the mappings $p(\zeta)$ and $\lambda(\zeta)$ for typical parameter values in the genus-2 case are shown in figure 3. Similar plots are shown for the genus-3 case in figure 4.

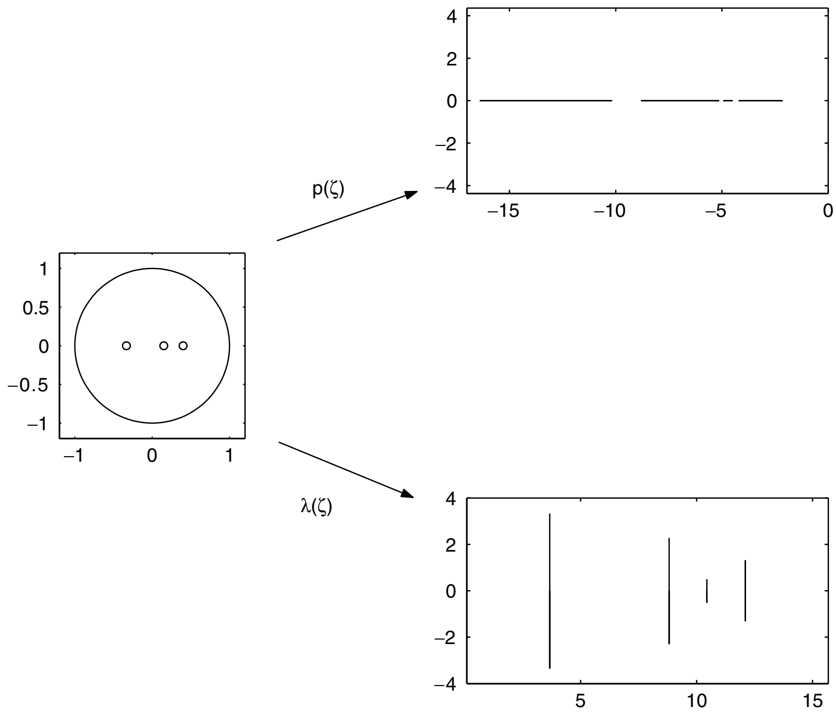

Figure 4. An illustration of the first type of reduction in the genus-3 case. The figures show the images of the annulus under the conformal mappings (66) with parameter values $\alpha = -0.15$, $q_1 = q_2 = q_3 = 0.05$, $\delta_1 = 0.4$, $\delta_2 = 0.15$ and $\delta_3 = -1/3$.

8.2. Second-type reductions

On use of (11) and (61), the solutions for $p(\zeta)$ and $\lambda(\zeta)$ for the second type of reduction are given by

$$\begin{aligned} p(\zeta) &= -\frac{1}{\alpha}\left(\alpha\frac{\omega_\alpha(\zeta,\alpha)}{\omega(\zeta,\alpha)} - \frac{1}{\alpha}\frac{\omega_\alpha(\zeta,\alpha^{-1})}{\omega(\zeta,\alpha^{-1})}\right), \\ \lambda(\zeta) &= \tilde{A}\left(\frac{\omega(\zeta,\beta)\omega(\zeta,\alpha^{-1})}{\omega(\zeta,\alpha)\omega(\zeta,\beta^{-1})}\right) + \tilde{C}, \end{aligned} \tag{66}$$

where $\tilde{A}$ and $\tilde{C}$ are real constants and we have taken $\bar{\alpha} = \alpha$ and $\bar{\beta} = \beta$. Some algebra, from local expansions of the above functions about $\zeta = \alpha$, reveals that

$$\begin{aligned} \tilde{A} &= \frac{\omega'(\alpha,\alpha)\omega(\alpha,\beta^{-1})}{\omega(\alpha,\beta)\omega(\alpha,\alpha^{-1})}, \\ \tilde{C} &= -\frac{\omega'_\alpha(\alpha,\alpha)}{\omega'(\alpha,\alpha)} + \frac{1}{\alpha^2}\frac{\omega_\alpha(\alpha,\alpha^{-1})}{\omega(\alpha,\alpha^{-1})} - \tilde{A}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\omega(\zeta,\beta)\omega(\zeta,\alpha^{-1})}{\omega'(\zeta,\alpha)\omega(\zeta,\beta^{-1})}\right)\Bigg|_{\zeta=\alpha}. \end{aligned} \tag{67}$$

Figure 5 shows the images of D_ζ under the conformal mappings $p(\zeta)$ and $\lambda(\zeta)$ for some typical parameter values in the genus-2 case. Figure 6 shows a typical genus-3 case.

Finally, we mention how the above diagrams were constructed. It is necessary, for a numerical implementation, to truncate the infinite product defining the prime function (19). This is done in a natural way by including all maps in the set Θ'' up to some chosen *level*. For example, the level-2 maps are all those mappings consisting of all compositions of any two

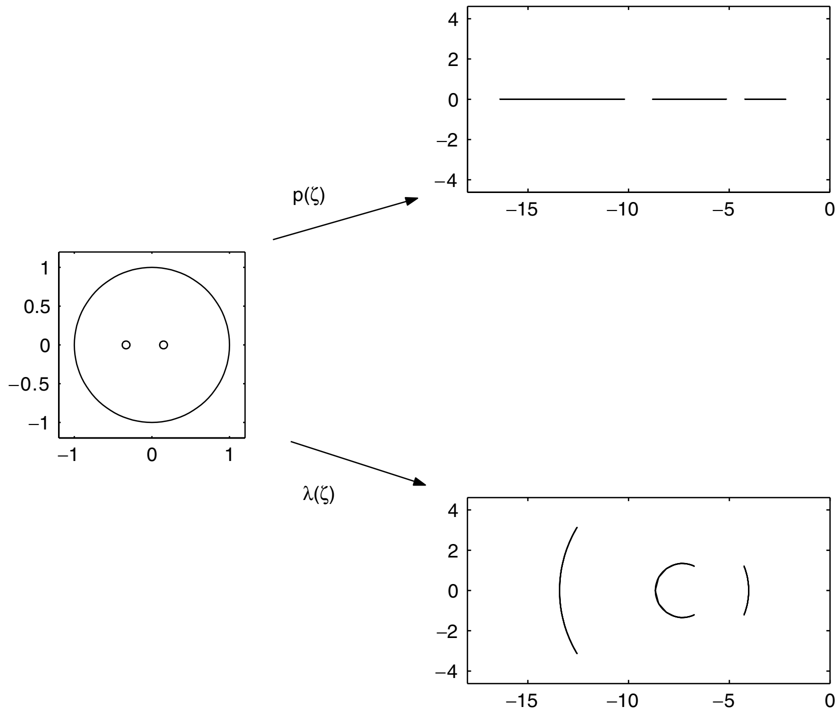

Figure 5. An illustration of the second type of reduction in the genus-2 case. The figures show the images of the annulus under the conformal mappings (64) with parameter values $\alpha = -0.15$, $\beta = 0.8$, $q_1 = q_2 = 0.05$, $\delta_1 = 0.15$ and $\delta_2 = -1/3$.

of the generating maps that do not reduce to the identity. The diagrams here were prepared by truncating the product at level 4 (i.e. by using all maps up to and including the level-3 maps). It is clear that, provided the moduli of the group are such that the prime function converges, the Schottky uniformization illustrated above provides a useful and practicable tool in uniformizing the spectral problem. Furthermore, in terms of it, the modified Green's function (61) takes an attractively simple functional form.

9. Discussion

The first reduction type considered here provides a reappraisal of those reductions already constructed in [25–27]. In the genus-2 case, for example, Baldwin and Gibbons [27] find formulae for p and λ by considering the associated hyperelliptic curve and inverting a second-kind Abelian integral on its Θ-divisor [26]. In the higher genus case, the same construction is performed on a stratum of the Jacobi variety of the relevant hyperelliptic curves [27]. The present work is an alternative to these constructions and yields different, but related, formulae. An attractive feature of our construction is that many of the conditions that must be explicitly imposed during the construction given in [25–27] are implicitly and automatically enforced by the general formulae (10) and (11) expressed in terms of modified Green's functions.

Of course, while we have elected to use the Schottky model to construct the modified Green's functions, other constructions/representations of these functions can be used in conjunction with formulae (10) and (11). A representation of the modified Green's function in

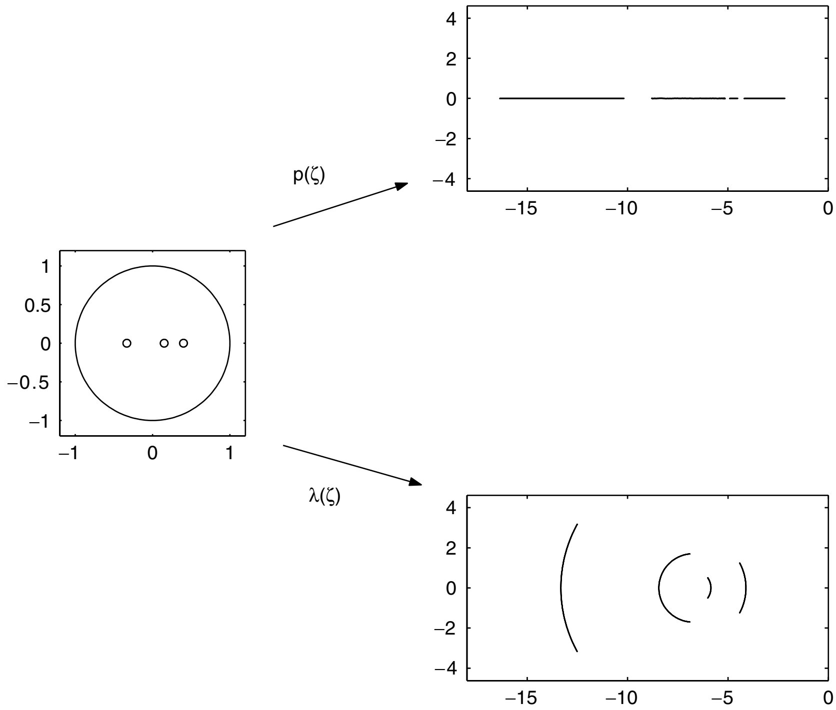

Figure 6. An illustration of the second type of reduction in the genus-3 case. The figures show the images of the annulus under the conformal mappings (66) with parameter values $\alpha = -0.15$, $\beta = 0.6$, $q_1 = q_2 = q_3 = 0.05$, $\delta_1 = 0.4$, $\delta_2 = 0.15$ and $\delta_3 = -1/3$.

terms of the prime form on the Schottky double expressed using Riemann theta functions [32] is an obvious alternative. Presumably then, (10) will reproduce the formulae, derived using very different arguments, of Baldwin and Gibbons [27] in the genus-N hyperelliptic case (the first-type reductions of this paper). It would be of interest to confirm this.

The new formulation here also suggests a reappraisal of reductions of the Benney hierarchy as a special class of flows in the extended moduli space of analytic curves. This mirrors recent work in [30, 31] where an identification between exact solutions of the equations governing Laplacian growth are reinterpreted as special reductions, known as 'algebraic orbits', of the universal Whitham hierarchy. In the latter case, the flows are generated by certain meromorphic differentials defined on the Schottky double of a multiply connected planar domain (the associated domains are known as *quadrature domains* [4, 9]). Here, with respect to the two reduction types of the Benney hierarchy just considered, the generating differentials $\mathrm{d}p$ and $\mathrm{d}\lambda$ are both second-kind Abelian differentials with two second-order poles (of vanishing residue), one on each half of the Schottky double of the planar domain D_ζ.

It is interesting that it was the association with conformal mappings and the dispersionless Toda hierarchy [28, 29] that first led to the association of the Dirichlet problem with the universal Whitham hierarchy [30–32]. Here it is again conformal mappings that have been the signpost to a theoretical connection between the Benney hierarchy with the Dirichlet problem in planar domains.

With such a convenient representation of the solutions to the spectral problem at hand, it is natural now to investigate whether it can be used to facilitate the calculation of the dynamical evolution of the system. Work on this interesting problem is currently in progress.

Acknowledgment

The author acknowledges many useful discussions over the years with his colleague Dr John Gibbons who first introduced him to the mathematics of the Benney hierarchy.

Appendix. Properties of G_j on the boundary circles

In this appendix, properties of the Schottky–Klein prime function are used to establish the properties of the functions $G_j(\zeta;\alpha)$ given in (31).

First, consider the complex conjugate of $R_j(\zeta;\alpha)$ as defined in (34) for ζ on C_j. There,

$$\overline{R_j(\zeta;\alpha)} = \frac{\overline{\omega}(\bar{\zeta},\bar{\alpha})\omega(\overline{\phi}_j(\bar{\zeta}),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\bar{\zeta},\phi_j(\alpha))\omega(\overline{\phi}_j(\bar{\zeta}),\alpha)}. \tag{A.1}$$

But, for ζ on C_j,

$$\bar{\zeta} = \phi_j(\zeta), \tag{A.2}$$

which implies that, on C_j,

$$\zeta = \overline{\phi}_j(\phi_j(\zeta)), \tag{A.3}$$

so that

$$\overline{R_j(\zeta;\alpha)} = \frac{\overline{\omega}(\phi_j(\zeta),\bar{\alpha})\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))\omega(\zeta,\alpha)} = \frac{1}{R_j(\zeta;\alpha)}. \tag{A.4}$$

This confirms that

$$|R_j(\zeta;\alpha)| = 1, \qquad \text{on} \quad C_j. \tag{A.5}$$

Consider next points ζ on C_0. There,

$$\begin{aligned}\overline{R_j(\zeta;\alpha)} &= \frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})\omega(\overline{\phi}_j(\zeta^{-1}),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))\omega(\overline{\phi}_j(\zeta^{-1}),\alpha)} \\ &= \frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})\omega(\theta_j(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))\omega(\theta_j(\zeta),\alpha)},\end{aligned} \tag{A.6}$$

where we have used the fact that $\bar{\zeta} = \zeta^{-1}$ for ζ on C_0 and the identity $\overline{\phi}_j(\zeta^{-1}) = \theta_j(\zeta)$. But

$$\frac{\omega(\theta_j(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_j(\zeta),\alpha)} = \beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)\frac{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\omega(\zeta,\alpha)}, \tag{A.7}$$

where we have used (23), and

$$\frac{\overline{\omega}(\overline{\theta}_j(\zeta^{-1}),\bar{\alpha})}{\overline{\omega}(\overline{\theta}_j(\zeta^{-1}),\phi_j(\alpha))} = \overline{\beta}_j(\bar{\alpha},\phi_j(\alpha))\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))}, \tag{A.8}$$

which implies that

$$\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))} = \frac{1}{\overline{\beta}_j(\bar{\alpha},\phi_j(\alpha))}\frac{\overline{\omega}(\phi_j(\zeta),\bar{\alpha})}{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}. \tag{A.9}$$

On use of (A.7) and (A.9) in (A.6), we get

$$\overline{R_j(\zeta;\alpha)} = \frac{\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)}{\overline{\beta}_j(\bar{\alpha},\phi_j(\alpha))}\frac{1}{R_j(\zeta;\alpha)} \tag{A.10}$$

or

$$|R_j(\zeta;\alpha)| = |\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)|. \tag{A.11}$$

Finally, consider points ζ on C_k where $k \neq j$. There,

$$\begin{aligned}
\overline{R_j(\zeta;\alpha)} &= \frac{\overline{\omega}(\phi_k(\zeta),\bar{\alpha})\omega(\overline{\phi}_j(\phi_k(\zeta)),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\phi_k(\zeta),\phi_j(\alpha))\omega(\overline{\phi}_j(\phi_k(\zeta)),\alpha)} \\
&= \frac{\overline{\omega}(\overline{\theta}_k(\zeta^{-1}),\bar{\alpha})}{\overline{\omega}(\overline{\theta}_k(\zeta^{-1}),\phi_j(\alpha))}\frac{\omega(\theta_j((\phi_k(\zeta))^{-1}),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_j((\phi_k(\zeta))^{-1}),\alpha)} \\
&= \beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha))\overline{\beta}_k(\bar{\alpha},\phi_j(\alpha))\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))}\frac{\omega((\phi_k(\zeta))^{-1},\overline{\phi}_j(\bar{\alpha}))}{\omega((\phi_k(\zeta))^{-1},\alpha)} \\
&= \beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha))\overline{\beta}_k(\bar{\alpha},\phi_j(\alpha))\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))}\frac{\omega\big(\theta_k^{-1}(\zeta),\overline{\phi}_j(\bar{\alpha})\big)}{\omega\big(\theta_k^{-1}(\zeta),\alpha\big)},
\end{aligned} \tag{A.12}$$

where, in the last equation, we have made use of (17). But,

$$\frac{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\omega(\zeta,\alpha)} = \beta_k(\overline{\phi}_j(\bar{\alpha}),\alpha)\frac{\omega\big(\theta_k^{-1}(\zeta),\overline{\phi}_j(\bar{\alpha})\big)}{\omega(\theta_k^{-1}(\zeta),\alpha)}. \tag{A.13}$$

On use of (A.9) and (A.13) in (A.12),

$$\overline{R_j(\zeta;\alpha)} = \frac{\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)\overline{\beta}_k(\bar{\alpha},\phi_j(\alpha))}{\overline{\beta}_j(\bar{\alpha},\phi_j(\alpha))\beta_k(\overline{\phi}_j(\bar{\alpha}),\alpha)}\frac{1}{R_j(\zeta;\alpha)} \tag{A.14}$$

or

$$|R_j(\zeta;\alpha)| = \left|\frac{\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi}_j(\bar{\alpha}),\alpha)}\right|. \tag{A.15}$$

To summarize all the above results, we conclude that

$$|R_j(\zeta;\alpha)| = \left|\frac{\beta_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right|, \qquad \text{on} \quad C_k. \tag{A.16}$$

Formula (A.16) holds for all j and k provided we adopt the convention that $\beta_0(\zeta,\alpha) \equiv 1$.

References

[1] Benney D J 1973 Some properties of long nonlinear waves *Stud. Appl. Math.* **52** 45

[2] Ablowitz M J and Fokas A S 1997 *Complex Variables* (Cambridge: Cambridge University Press)

[3] Crowdy D G 2005 The Benney hierarchy and the Dirichlet boundary problem in two dimensions *Phys. Lett.* A **343** 319–29

[4] Crowdy D G and Marshall J S 2004 The construction of multiply connected quadrature domains *SIAM J. Appl. Math.* **64** 1334–59

[5] Crowdy D G and Marshall J S 2005 Explicit formulae for the Kirchhoff–Routh path function for point vortex motion in multiply connected domains *Proc. R. Soc.* A **461** 2477–501

[6] Richardson S 1972 Hele–Shaw flows with a free boundary produced by the injection of fluid into a narrow channel *J. Fluid Mech.* **56** 609–18

[7] Richardson S 2001 Hele–Shaw flows with time-dependent free boundaries involving a multiply connected fluid region *Eur. J. Appl. Math.* **12** 571–99

[8] Crowdy D G 2005 Quadrature domains and fluid dynamics *Operator Theory: Adv. Appl.* **156** 113–29

[9] Gustafsson B and Shapiro H S 2005 What is a quadrature domain? *Operator Theory: Adv. Appl.* **156** 1

[10] Baker H 1995 *Abelian Functions* (Cambridge: Cambridge University Press)

[11] Beardon A F 1984 *A Primer on Riemann Surfaces (London Math. Soc. Lecture Note Series* vol 78*)* (Cambridge: Cambridge University Press)

[12] Fricke R and Klein F 1897 *Vorlesungen uber die Theorie der automorphen Funktionen* vol 1 (Leipzig: Teubner)

[13] Belokolos E D, Bobenko A I, Enolskii V Z and Its A R 1994 *Algebro-Geometric Approach to Nonlinear Integrable Equations* (New York: Springer)

[14] Burnside W 1892 On a class of automorphic functions *Proc. Lond. Math. Soc.* **23** 49–88

[15] Burnside W 1892 Further note on automorphic functions *Proc. Lond. Math. Soc.* **23** 281–95

[16] Tsarev S P 1985 On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type *Sov. Math. Dokl.* **31** 488

[17] Koebe P 1914 Abhandlungen zur theorie der konformen abbildung *Acta Math.* **41** 305–44

[18] Mumford D, Series C and Wright D 2002 *Indra's Pearls* (Cambridge: Cambridge University Press)

[19] Etingof P I and Varchenko A 1992 *Why Does the Boundary of a Round Drop Become a Curve of Order Four University Lecture Series* vol 3*)* (Providence, RI: American Mathematical Society)

[20] Gustafsson B 1983 Quadrature identities and the Schottky double *Acta Appl. Math.* **1** 209–40

[21] Shapiro H 1992 *The Schwarz Function and its Generalization to Higher Dimensions University of Arkansas Lecture Notes in the Mathematical Sciences* vol 9*)* (New York: Wiley-Interscience)

[22] Nehari Z 1952 *Conformal Mapping* (New York: McGraw-Hill)

[23] Schiffer M 1950 Recent advances in the theory of conformal mapping. Appendix to Courant R *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces* (New York: Interscience)

[24] Gibbons J and Tsarev S P 1999 Conformal maps and reductions of the Benney equations *Phys. Lett.* A **258** 263–71

[25] Yu L and Gibbons J 2000 The initial value problem for reductions of the Benney equations *Inverse Problems* **16** 605–18

[26] Baldwin S and Gibbons J 2003 Hyperelliptic reduction of the Benney moment equations *J. Phys. A: Math. Gen.* **36** 8393

[27] Baldwin S and Gibbons J 2004 Higher genus hyperelliptic reduction of the Benney moment equations *J. Phys. A: Math. Gen.* **37** 5341–54

[28] Mineev-Weinstein M, Wiegmann P B and Zabrodin A 2000 Integrable structure of interface dynamics *Phys. Rev. Lett.* **84** 5106

[29] Wiegmann P B and Zabrodin A 2000 Conformal maps and integrable hierarchies *Commun. Math. Phys.* **213** 523–38

[30] Marshakov A, Wiegmann P B and Zabrodin A 2002 Integrable structure of the Dirichlet boundary problem in two dimensions *Commun. Math. Phys.* **227** 131–53

[31] Krichever I, Mineev-Weinstein M, Wiegmann P B and Zabrodin A 2004 Laplacian growth and Whitham equations of soliton theory *Physica* D **198** 1–28

[32] Krichever I, Marshakov A and Zabrodin A 2005 Integrable structure of the Dirichlet boundary problem in multiply connected domains *Commun. Math. Phys.* **259** 1–44

[33] Takhtajan L 2001 Free bosons and tau-functions for compact Riemann surfaces and closed smooth Jordan curves: I. Current correlation functions *Lett. Math. Phys.* **56** 181–228